PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Closed Equations of Motion for Expectation Values of Collective Operators for Spontaneous Emission in the Presence of External Driving Fields

J. Seke

Phil. Trans. R. Soc. Lond. A 1984 **313**, 449-451
doi: 10.1098/rsta.1984.0137

Phil Trans. R. Soc. Lond. A **313**, 449–451 (1984)
Printed in Great Britain

Closed equations of motion for expectation values of collective operators for spontaneous emission in the presence of external driving fields

By J. Seke

Institut für Theoretische Physik, Technische Universität Wien, Karlsplatz 13, *A*-1040 *Vienna, Austria*

By using a modified Robertson projection technique, developed by the author in previous work, three exact closed equations of motion for expectation values of collective atomic operators for spontaneous emission in the presence of a classical driving field are derived.

In this paper I shall derive exact closed equations of motion for spontaneous emission from a system of N two-level atoms in the presence of a strong classical driving field.

The modified Robertson projection-operator technique, developed in previous work (Seke 1980; Adam & Seke 1981), is based on the application of a special time-dependent projection-operator $P(t)$, which directly picks out the expectation values that are of interest. This projection operator transforms, by definition, the time derivative of the statistical density operator $\rho(t)$ of the total system S+R (S – system of N atoms with classical driving field and R – quantized electromagnetic field into which the atoms radiate) into the time derivative of $\sigma_{\mathrm{S}}(t)\otimes\rho_{\mathrm{R}}(0)$, where $\sigma_{\mathrm{S}}(t)$ is the generalized canonical density operator (Robertson 1966; Seke 1980) of system S and $\rho_{\mathrm{R}}(0)$ is the initial density operator of system R. It is assumed that systems S and R are statistically independent at $t=0$: $\rho(0)=\rho_{\mathrm{S}}(0)\otimes\rho_{\mathrm{R}}(0)$.

The Hamiltonian of the total system $H(t)=H_0(t)+H_{\mathrm{SR}}$ in the dipole and rotating-wave approximation can be written as

$$\begin{aligned} H_0(t) &= H_{\mathrm{S}}+H_{\mathrm{R}}+H_{\mathrm{ext}}(t) \\ &= \omega S^z+\sum_{k,s}\omega_k\, a^+_{ks}\, a^-_{ks}+(gS^+\,\mathrm{e}^{-\mathrm{i}\omega_0 t}+g^*S^-\,\mathrm{e}^{\mathrm{i}\omega_0 t}), \end{aligned} \tag{1}$$

$$H_{\mathrm{SR}}=\sum_{i,ks}(g_{iks}\,S_i^+\otimes a^-_{ks}+g^*_{iks}\,S_i^-\otimes a^+_{ks}), \tag{2}$$

where ω is the energy separation of the two atomic levels, ω_0 is the frequency of the harmonically varying external classical driving field (Mollow 1969; Agarwal 1974), $\omega_k=kc$ is the frequency of the kth mode of the quantized electromagnetic field, S_i^z and $S_i^\pm$ are the population inversion and dipole moment operators of the ith atom, $S^z\equiv\Sigma_i S_i^z$ and $S^\pm\equiv\Sigma_i S_i^\pm\,\mathrm{e}^{\pm\mathrm{i}k_0 r_i}$ ($\boldsymbol{r}_i$ is position vector of the ith atom) are the collective atomic operators, a^+_{ks} and a^-_{ks} are the photon creation and annihilation operators for the mode $\boldsymbol{k}s$ (s is the polarization index) and g and $g_{iks}=g_{ks}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}_i}$ are the coupling coefficients.

We now introduce a generalized canonical density operator for our collective operators

$$\sigma_{\mathrm{S}}(t)=\prod_{i=1}^{N}[\tfrac{1}{2}I_i+\sum_{\beta=(+,-,z)}\frac{d^\beta}{N}(S^\beta_{i,k_0})^+\langle S^\beta\rangle_t]=\prod_{i=1}^{N}\sigma_{\mathrm{S}}^{(i)}(t) \tag{3}$$

and corresponding time-dependent projection operator

$$P(t)\ldots = \rho_{\mathrm{R}}(0) \otimes \sum_{j=1}^{N} \prod_{\substack{i=1 \\ i \neq j}}^{N} \sigma_{\mathrm{S}}^{(i)}(t) \sum_{\beta=(+,-,z)} \frac{d^{\beta}}{N} (S^{\beta}_{j,k_0})^{+} \mathrm{Tr}_{\mathrm{SR}}(S^{\beta}\ldots), \tag{4}$$

where I_i is the unit operator in the ith factor space, $d^{\pm} = 1$, $d^{z} = 2$ and $S^{\pm}_{i,k_0} = S^{\pm}_{i}\, \mathrm{e}^{\pm \mathrm{i}k_0 r_i}$, $S^{z}_{i,k_0} = S^{z}_{i}$.

Our set of operators $\{S^{\beta}\}$ is chosen so that the commutator $[H_0(t), S^{\beta}]$ can be expressed as a linear combination of the operators of the set and so that $\sigma_{\mathrm{S}}(t)$ can describe the initial conditions with uncorrelated and permutationally symmetric atoms. In the following calculations I will treat only the special initial conditions

$$\rho(0) = \sigma_{\mathrm{S}}(0) \otimes \rho_{\mathrm{R}}(0) \tag{5}$$

with $\rho_{\mathrm{R}}(0) = |\{0\}\rangle\langle\{0\}|$ being the vacuum radiation state.

By using the Robertson projection-operator formalism (Robertson 1966) from the equation

$$[\mathrm{d}\sigma_{\mathrm{S}}(t)/\mathrm{d}t] \otimes \rho_{\mathrm{R}}(0) = (-\mathrm{i})\, P(t)\, L(t)\, \rho(t), \tag{6}$$

we obtain the connecting equation between $\rho(t)$ and $\sigma_{\mathrm{S}}(t) \otimes \rho_{\mathrm{R}}(0)$:

$$\rho(t) - \sigma_{\mathrm{S}}(t) \otimes \rho_{\mathrm{R}}(0) = -\mathrm{i} \int_0^t \mathrm{d}t'\, U_{\mathrm{I}}(t,t')\, U_0(t,t')\, [I - P(t')]\, L(t')\, \sigma_{\mathrm{S}}(t') \otimes \rho_{\mathrm{R}}(0), \tag{7}$$

where

$$U_{\mathrm{I}}(t,t') = T \exp\left\{-\mathrm{i} \int_{t'}^{t} \mathrm{d}t_1\, U_0(t,t_1)\, [I - P(t_1)]\, L_{\mathrm{SR}}\, U_0^{-1}(t,t_1)\right\}, \tag{8}$$

$$U_0(t,t') = T \exp\left[-\mathrm{i} \int_{t'}^{t} \mathrm{d}t_1\, L_0(t_1)\right], \tag{9}$$

with T as the Dyson time-ordering operator and $L_0(t)$, L_{S}, L_{R}, $L_{\mathrm{ext}}(t)$, L_{SR} as the Liouvillians corresponding to the Hamiltonians in (1) and (2).

If we let the operator $\mathrm{i}[(L_{\mathrm{S}} + L_{\mathrm{ext}}(t))\, S^{\beta}]$ act on (7) and afterwards take the trace over it, we obtain a set of *exact closed* equations of motion for expectation values of collective atomic operators S^{β}:

$$\begin{aligned} \mathrm{d}\langle S^{\beta}\rangle_t/\mathrm{d}t &= \mathrm{i}\, \mathrm{Tr}_{\mathrm{S}}\{[(L_{\mathrm{S}} + L_{\mathrm{ext}}(t))\, S^{\beta}]\, \sigma_{\mathrm{S}}(t)\} \\ &\quad + \int_0^t \mathrm{d}t'\, \mathrm{Tr}_{\mathrm{SR}}\, [(L_{\mathrm{SR}}\, S^{\beta})\, U_{\mathrm{I}}(t,t')\, U_0(t,t')\, L_{\mathrm{SR}}\, \sigma_{\mathrm{S}}(t') \otimes \rho_{\mathrm{R}}(0)], \quad \beta = (+,-,z). \end{aligned} \tag{10}$$

The equations of motion obtained are closed since $U_{\mathrm{I}}(t,t')$, $P(t)$ and $\sigma_{\mathrm{S}}(t)$ are functions of the expectation values $\langle S^{\beta}\rangle_t$, $\beta = (+,-,z)$. These equations can easily be transformed to the rotating frame because the operators $\sigma_{\mathrm{S}}(t)$ and $P(t)$ are invariant under such a transformation.

For a single atom

$$\sigma_{\mathrm{S}}(t) = \tfrac{1}{2} I + \sum_{\beta} \mathrm{d}^{\beta} (S^{\beta})^{+} \langle S^{\beta}\rangle_t,$$

$$P(t) = P = \rho_{\mathrm{R}}(0) \otimes \sum_{\beta} \mathrm{d}^{\beta} (S^{\beta})^{+}\, \mathrm{Tr}_{\mathrm{SR}}(S^{\beta}\ldots),$$

and the equations of motion become much simpler. In the Markov and long-time approximations (but without making the Born approximation), these equations reduce to three linear differential equations.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Here we have shown that it is possible to derive three *exact closed* equations of motion for collective expectation values without making any restriction as to the dimensions of the atomic system. The derivation of such equations of motion is only possible by using our modified Robertson projection technique, because this method takes into account the special initial density operator of uncorrelated and permutationally symmetric atoms during the whole calculation. These special initial conditions have the consequence that all the multi-atom correlations can be expressed as functionals of the expectation values $\langle S^{\beta}\rangle_t$:

$$\langle S^{\beta_1}_{i_1,k_0} S^{\beta_2}_{i_2,k_0} \dots S^{\beta_n}_{i_n,k_0}\rangle_t - \left(\frac{1}{N}\right)^n \langle S^{\beta_1}\rangle_t \langle S^{\beta_2}\rangle_t \dots \langle S^{\beta_n}\rangle_t$$
$$= -\mathrm{i}\int_0^t \mathrm{d}t' \,\mathrm{Tr}_{\mathrm{SR}}\,[S^{\beta_1}_{i_1,k_0} S^{\beta_2}_{i_2,k_0} \dots S^{\beta_n}_{i_n,k_0} U_{\mathrm{I}}(t,t')\, U_0(t,t')\, L_{\mathrm{SR}}\, \sigma_{\mathrm{S}}(t') \otimes \rho_{\mathrm{R}}(0)],$$
$$n \leqslant N, i_r \neq i_s \,\forall\, r \neq s, \quad (11)$$

as can easily be seen from (7).

Contrary to this, other methods that use the Zwanzig time-independent projection-operator technique (Zwanzig 1960; Emch & Sewell 1968; Agarwal 1974) cannot take into account the special initial conditions with regard to the system S and the derived master equations hold for quite general initial density operators $\rho_{\mathrm{S}}(0)$. This is the reason why, by using these methods, it is not possible to derive closed equations of motion, but a hierarchy of equations containing all the expectation values $\langle S^{\beta_1} S^{\beta_2} \dots S^{\beta_n}\rangle_t$, $\beta_1, \dots, \beta_n = (+, -, z)$, $(n \leqslant N)$. But this can only be achieved for geometrically small atomic systems (Agarwal 1974) and not for arbitrary large systems like those discussed here.

References

Adam, G. & Seke, J. 1981 *Phys. Rev.* A **23**, 3118.
Agarwal, G. S. 1974 *Springer Tracts mod. Phys.* **70**.
Emch, G. G. & Sewell, G. L. 1968 *J. math. Phys.* **9**, 946.
Mollow, B. R. 1969 *Phys. Rev.* **188**, 1969.
Robertson, B. 1966 *Phys. Rev.* **144**, 151.
Seke, J. 1980 *Phys. Rev.* A **21**, 2156.
Zwanzig, R. 1960 *J. chem. Phys.* **33**, 1338.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES